



Examen Parcial – 10/05/11

1. Un cilindro de eje paralelo al eje x_3 y cuya sección normal es el círculo de radio a está sometido a torsión por cuplas que actúan en los extremos $x_3 = -L$ y $x_3 = L$. Se asume que las componentes de tensión están dadas por:

$$\sigma_{13} = -\frac{\partial \psi}{\partial x_2}, \quad \sigma_{23} = \frac{\partial \psi}{\partial x_1}, \quad \sigma_{11} = \sigma_{12} = \sigma_{22} = \sigma_{33} = 0$$

donde $\psi = \psi(x_1, x_2)$ es una función conocida.

- Mostrar que, para este sistema sin cargas de volumen, el tensor de tensiones verifica la ecuación de equilibrio (se dice que el tensor está *autoequilibrado*).
- Mostrar algebraicamente que la diferencia entre la máxima tensión principal y la mínima tensión principal es $2 \sqrt{\left(\frac{\partial \psi}{\partial x_1}\right)^2 + \left(\frac{\partial \psi}{\partial x_2}\right)^2}$. Mostrar que la segunda tensión principal es nula y hallar el eje principal correspondiente a ésta.
- Para el caso particular $\psi = r^2 = x_1^2 + x_2^2$, mostrar que en la frontera del cilindro ($r = a$, y $x_3 = \pm L$), la componente normal del vector de tracción es nula. Mostrar además que en la superficie lateral $r = a$, la componente de corte del vector de tracción es nula, en tanto en los extremos la misma toma valores no nulos.
- También para el caso $\psi = r^2 = x_1^2 + x_2^2$, mostrar que en cada cara de extremidad ($x_3 = \pm L$), la cupla externa actuante es $\pm \pi a^4$.

Ayuda: **mostrar** que la cupla actuando sobre una cara de extremidad es igual a la integral sobre el área del momento generado por las componentes de corte del vector de tracción, o sea:

$$M = \int_A (\sigma_{32} x_1 - \sigma_{31} x_2) dA$$

2. Sea \mathbf{r} un radio vector y r su magnitud. Probar usando notación indicial (siendo n un número entero):

$$\nabla \times (r^n \mathbf{r}) = 0$$

3. Determinar las direcciones principales y los valores principales del tensor Cartesiano de segundo orden \mathbf{T} , cuya representación matricial es la siguiente:

$$T = \begin{bmatrix} 6 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 6 \end{bmatrix}$$

Resolver algebraicamente y representar la solución usando el círculo de Mohr. Mostrar que los ejes principales calculados forman un conjunto de ejes ortogonales

1) a) $\nabla_{ij,j} + X_i = 0$

No hay cargas de volumen $\rightarrow X_i = 0$

$$\nabla_{11,1} + \nabla_{12,2} + \nabla_{13,3} =$$

$$0 + 0 + \underbrace{\left(-\frac{\partial^2 \Psi}{\partial x_2 \partial x_3}\right)}_0 = 0$$

$$\nabla_{21,1} + \nabla_{22,2} + \nabla_{23,3} =$$

$$0 + 0 + \underbrace{\left(\frac{\partial^2 \Psi}{\partial x_1 \partial x_3}\right)}_0 = 0$$

$$\nabla_{31,1} + \nabla_{32,2} + \nabla_{33,3} =$$

$$\left(-\frac{\partial^2 \Psi}{\partial x_2 \partial x_1}\right) + \frac{\partial^2 \Psi}{\partial x_1 \partial x_2} + 0 = 0$$

QED

b)

$$\underline{\underline{\nabla}} = \begin{bmatrix} 0 & 0 & -\frac{\partial \Psi}{\partial x_2} \\ 0 & 0 & \frac{\partial \Psi}{\partial x_1} \\ -\frac{\partial \Psi}{\partial x_2} & \frac{\partial \Psi}{\partial x_1} & 0 \end{bmatrix}$$

$$\det(\underline{\underline{\nabla}} - \nabla^k \underline{\underline{I}}) = 0 \quad k = 1, 2, 3$$

$$\det \begin{pmatrix} -\nabla & 0 & -\frac{\partial \Psi}{\partial x_2} \\ 0 & -\nabla & \frac{\partial \Psi}{\partial x_1} \\ -\frac{\partial \Psi}{\partial x_2} & \frac{\partial \Psi}{\partial x_1} & -\nabla \end{pmatrix} = 0$$

(2)

$$\begin{aligned}
 & (-\sigma) \left[\sigma^2 - \left(\frac{\partial \psi}{\partial x_1} \right)^2 \right] + \left(-\frac{\partial \psi}{\partial x_2} \right) \left(-\sigma \frac{\partial \psi}{\partial x_2} \right) = \\
 & = -\sigma^3 + \sigma \left(\frac{\partial \psi}{\partial x_1} \right)^2 + \sigma \left(\frac{\partial \psi}{\partial x_2} \right)^2 = \\
 & = \sigma \left[\left(\frac{\partial \psi}{\partial x_1} \right)^2 + \left(\frac{\partial \psi}{\partial x_2} \right)^2 - \sigma^2 \right] = 0
 \end{aligned}$$

$$\begin{aligned}
 \therefore \sigma &= \begin{pmatrix} 0 \\ \sqrt{\left(\frac{\partial \psi}{\partial x_1} \right)^2 + \left(\frac{\partial \psi}{\partial x_2} \right)^2} \\ -\sqrt{\left(\frac{\partial \psi}{\partial x_1} \right)^2 + \left(\frac{\partial \psi}{\partial x_2} \right)^2} \end{pmatrix} = \sigma^2 \\
 & = \sigma^3 \\
 & = \sigma^1
 \end{aligned}$$

$$\sigma^3 - \sigma^1 = 2 \sqrt{\left(\frac{\partial \psi}{\partial x_1} \right)^2 + \left(\frac{\partial \psi}{\partial x_2} \right)^2}$$

$\sigma^2 = 0$			ϕ_1^2	Q.E.D.
			ϕ_2^2	
ϕ_3^2				
0	0	$-\frac{\partial \psi}{\partial x_2}$	0	
0	0	$\frac{\partial \psi}{\partial x_1}$	0	
$-\frac{\partial \psi}{\partial x_2}$	$\frac{\partial \psi}{\partial x_1}$	0	0	

$$\therefore -\left(\frac{\partial \psi}{\partial x_2} \right) \phi_3^2 = 0 \implies \phi_3^2 = 0$$

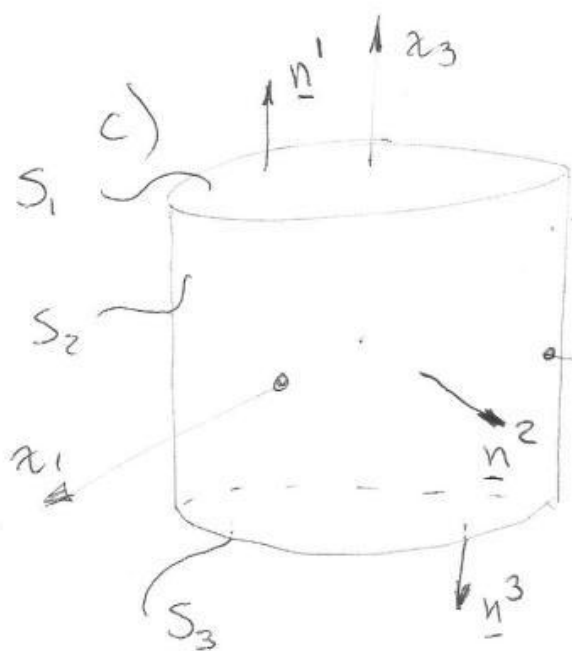
$$-\frac{\partial \psi}{\partial x_2} \phi_1^2 + \frac{\partial \psi}{\partial x_1} \phi_2^2 = 0$$

3

Elijo $\phi^2 = \frac{\partial \Psi}{\partial x_1} \Rightarrow \phi^2 = \frac{\partial \Psi}{\partial x_2}$

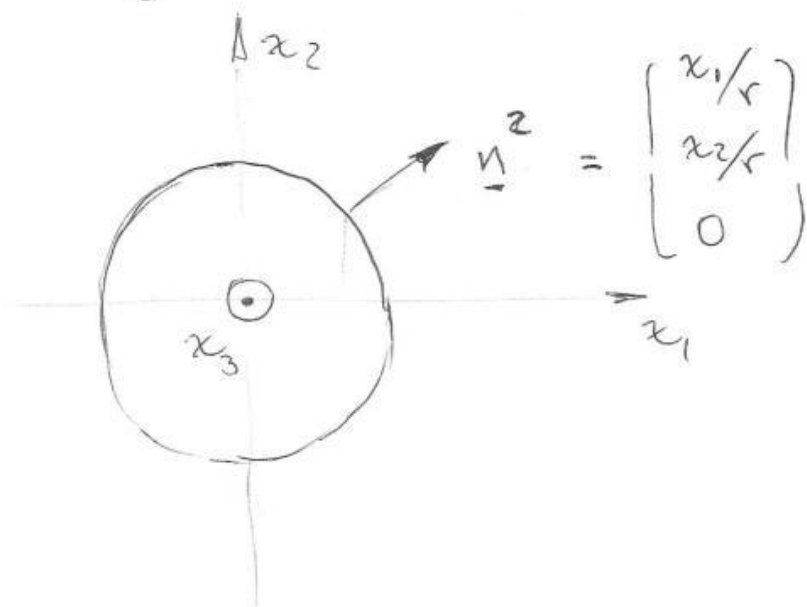
$$\phi^2 = \begin{pmatrix} \frac{\partial \Psi}{\partial x_1} \\ \frac{\partial \Psi}{\partial x_2} \\ 0 \end{pmatrix}$$

QED



$$\underline{n}^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{n}^3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$



(4)

$$\underline{t} = \underline{v} = \underline{n}$$

Tesis:

$$\underline{t}^{1,3} = \begin{bmatrix} 0 & 0 & -\frac{\partial \Psi}{\partial x_2} \\ 0 & 0 & \frac{\partial \Psi}{\partial x_1} \\ -\frac{\partial \Psi}{\partial x_2} & \frac{\partial \Psi}{\partial x_1} & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix} = \begin{pmatrix} \mp \frac{\partial \Psi}{\partial x_2} \\ \pm \frac{\partial \Psi}{\partial x_1} \\ 0 \end{pmatrix}$$

Comp normal:

$$\underline{t}_n^{1,3} = \underline{t}^{1,3} \cdot \underline{n}^{1,3} = \begin{pmatrix} \mp \frac{\partial \Psi}{\partial x_2} \\ \pm \frac{\partial \Psi}{\partial x_1} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix} = 0$$

Sup lateral:

$$\Psi = x_1^2 + x_2^2 \Rightarrow \frac{\partial \Psi}{\partial x_1} = 2x_1 \quad \frac{\partial \Psi}{\partial x_2} = 2x_2$$

\therefore

$$\underline{t}^2 = \begin{bmatrix} 0 & 0 & -2x_2 \\ 0 & 0 & 2x_1 \\ -2x_2 & 2x_1 & 0 \end{bmatrix} \begin{pmatrix} x_1/r \\ x_2/r \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

\therefore Comp normal y de corte son nulas.

(5)

Componentes de corte en toros:

$$\underline{t}^1 = \begin{pmatrix} -\frac{\partial \Psi}{\partial x_2} \\ \frac{\partial \Psi}{\partial x_1} \\ 0 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ 2x_1 \\ 0 \end{pmatrix}$$

NO NULAS

$$\underline{t}^2 = \begin{pmatrix} \frac{\partial \Psi}{\partial x_2} \\ -\frac{\partial \Psi}{\partial x_1} \\ 0 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ -2x_1 \\ 0 \end{pmatrix}$$

QED

d) Curls:

$$M = \int_A (\sigma_{32} x_1 - \sigma_{31} x_2) dA =$$

$$= \int_A \left(\frac{\partial \Psi}{\partial x_1} x_1 + \frac{\partial \Psi}{\partial x_2} x_2 \right) dA =$$

$$= \int_A (2x_1^2 + 2x_2^2) dA = \int_A 2r^2 dA =$$

$$= \int_0^{2\pi} \int_0^a 2r^2 r dr d\theta =$$

$$= 2\pi \left. \frac{2r^4}{4} \right|_0^a = \underline{\underline{\pi a^4}} \quad \text{QED}$$

$$2) \quad \nabla \times (r^n \underline{r}) = 0 \quad (6)$$

$$\underline{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\begin{aligned} & \epsilon_{ijk} \frac{\partial}{\partial x_j} (r^n \cdot k) = \\ & = \epsilon_{ijk} \frac{\partial r^n}{\partial x_j} x_k + \epsilon_{ijk} r^n \frac{\partial x_k}{\partial x_j} = \\ & = \epsilon_{ijk} n r^{n-1} \frac{\partial r}{\partial x_j} x_k + \underbrace{\epsilon_{ijk} r^n \delta_{jk}}_{\epsilon_{ikk} r^n = 0} \quad (*) \end{aligned}$$

$$\frac{\partial r}{\partial x_j} = \frac{x_j}{r} = \frac{x_j}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$(*) = \epsilon_{ijk} n r^{n-1} \frac{x_j}{r} x_k = n r^{n-2} \underbrace{\epsilon_{ijk} x_j x_k}_0$$

$$\boxed{\nabla \times (r^n \underline{r}) = 0} \quad \text{Q.E.D.}$$

(7)

$$3) \det \begin{pmatrix} 6-\sqrt{v} & 0 & -2 \\ 0 & -\sqrt{v} & 0 \\ -2 & 0 & 6-\sqrt{v} \end{pmatrix} = 0$$

$$\begin{aligned} & (6-\sqrt{v})(-\sqrt{v})(6-\sqrt{v}) + (-2)(-2\sqrt{v}) = \\ & = -\sqrt{v}(6-\sqrt{v})^2 + 4\sqrt{v} = \\ & = \sqrt{v}[-(6-\sqrt{v})^2 + 4] = \\ & = \sqrt{v}[-36 + 12\sqrt{v} - \sqrt{v}^2 + 4] = \\ & = \sqrt{v}[-\sqrt{v}^2 + 12\sqrt{v} - 32] = 0 \end{aligned}$$

$$\sqrt{v}^1 = 0$$

$$\sqrt{v}^{2,3} = \frac{-12 \pm \sqrt{12^2 - 4 \times 32}}{-2} =$$

$$= 6 \pm \frac{\sqrt{144 - 128}}{-2} =$$

$$= 6 \pm \frac{\sqrt{16}}{-2} = 6 \pm (-2) = \begin{cases} 4 \\ 8 \end{cases}$$

$$\sqrt{v}^1 = 0 \quad \sqrt{v}^2 = 4 \quad \sqrt{v}^3 = 8$$

$$\begin{array}{ccc|c} \phi_1' & & & \phi_1' \\ & & & \phi_2' \\ & & & \phi_3' \\ \hline 6 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 6 & 0 \end{array} \quad (8)$$

$$\begin{aligned} 6\phi_1' - 2\phi_3' &= 0 \\ -2\phi_1' + 6\phi_3' &= 0 \end{aligned} \quad \Rightarrow \quad \phi_1' = \phi_3' = 0$$

Elijo $\phi_2' = 1$

$$\therefore (\underline{v}, \underline{\phi}') = \left(0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\begin{array}{ccc|c} \phi_2' & & & \phi_1'' \\ & & & \phi_2'' \\ & & & \phi_3'' \\ \hline 2 & 0 & -2 & 0 \\ 0 & -4 & 0 & 0 \\ -2 & 0 & 2 & 0 \end{array}$$

$$-4\phi_2'' = 0 \quad \Rightarrow \quad \phi_2'' = 0$$

$$2\phi_1'' - 2\phi_3'' = 0 \quad \Rightarrow \quad \phi_1'' = \phi_3''$$

$$-2\phi_1'' + 2\phi_3'' = 0 \quad \Rightarrow \quad \phi_1'' = \phi_3''$$

$$\text{Elijo } \phi_1^2 = \phi_3^2 = 1$$

(9)

$$\therefore (\underline{v}^2, \underline{\phi}^2) = \left(4, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right)$$

$$\begin{array}{ccc|c} \phi^3) & & & \phi_1^3 \\ & & & \phi_2^3 \\ & & & \phi_3^3 \\ \hline -2 & 0 & -2 & 0 \\ 0 & -8 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}$$

$$-8 \phi_2^3 = 0 \longrightarrow \phi_2^3 = 0$$

$$-2 \phi_1^3 - 2 \phi_3^3 = 0 \longrightarrow \phi_1^3 = -\phi_3^3$$

$$\text{Elijo } \phi_1^3 = 1 \longrightarrow \phi_3^3 = -1$$

$$\therefore (\underline{v}^3, \underline{\phi}^3) = \left(8, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right)$$

Clasete:

$$\underline{\phi}^1 \cdot \underline{\phi}^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\underline{\phi}^1 \circ \underline{\phi}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

(10)

$$\underline{\phi}^2 \circ \underline{\phi}^3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

∴ Los ejes principales se ortogonales.

Notar que se trata de un problema plano en los ejes (x_2, x_3) , con

$$T = \begin{pmatrix} 6 & -2 \\ -2 & 6 \end{pmatrix}.$$

Luego: λ

